JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 88 (1997) 269–274

New hypergeometric identities arising from Gauss's second summation theorem

H. Exton

Nyuggel, Lunabister, Dunrossness, Shetland ZE2 9JH, UK

Received 16 June 1997; received in revised form 29 August 1997

Abstract

The elementary manipulation of series is applied to obtain a quite general transformation involving hypergeometric functions. A number of hypergeometric identities not previously recorded in the literature are then deduced from Gauss's second summation theorem and other hypergeometric summation theorems. © 1997 Elsevier Science B.V. All rights reserved.

Keywords: Generalized hypergeometric series; Gauss's second theorem; Vandermonde's theorem; Dixon's theorem; Kummer's theorem

AMS classification: 33C20; 68Q40

1. Introduction

Identities involving hypergeometric series have various applications including those associated with systems of computer-algebra manipulation. Summation formulae for hypergeometric series are applied to a general relation obtained by the elementary manipulation of series. In what follows, the Pochhammer symbol $(a,n) = \Gamma(a+n)/\Gamma(a)$ will be used, and any values of parameters leading to results which do not make sense are tacitly excluded. Indices of summation are taken to run over all of the nonnegative integers.

Consider the double series (assumed to be absolutely convergent or terminating):

$$S = \sum_{m,n} [C_{n+m}(a, 2n+m)x^{m+n}y^n]/[(b,n)m!n!], \tag{1.1}$$

where C_n is a generalized parameter.

PII S0377-0427(97)00189-1

Replace $n+m$ by N and obtain

$$S=\sum_{N,n}[C_N(a,n+N)x^N y^n]/[(b,n)n!(N-n)!]. \tag{1.2}$$

If (1.1) and (1.2) are equated and rearranged, we see that, after putting

$$C_n=[(c_1,n)\dots(c_C,n)]/[(d_1,n)\dots(d_D)], \tag{1.3}$$

we obtain, after a slight modification of the notation, the result

$$\sum_n [(c_1,n)\dots(c_C,n)(a,2n)x^n y^n]/[(d_1,n)\dots(d_D,n)(b,n)n!]$$
$$\times {}_{C+1}F_D[c_1+n,\dots,c_C+n,a+2n;d_1+n,\dots,d_D+n;x]$$
$$=\sum_n [(c_1,n)\dots(c_C,n)(a,n)x^n]/[(d_1,n)\dots(d_D,n)n!]{}_2F_1[a+n,-n;b;-y]. \tag{1.4}$$

The generalized hypergeometric function is given by

$${}_AF_B[a_1,\dots,a_A;b_1,\dots,b_B;x]=\sum_n [(a_1,n)\dots(a_A,n)x^n]/[(b_1,n)\dots(b_B,n)n!] \tag{1.5}$$

and it has been discussed at great length, including convergence properties, by Slater [2], Exton [1] and many other authors.

The transformation (1.4) is a special case of a formula given by Slater [2, p. 60], who obtained it by other means. The identities presented in this study are obtained from (1.4) by expressing the inner hypergeometric functions in closed form.

If we put $y=-\frac{1}{2}$ and $b=\frac{1}{2}+a/2$, then Gauss's second summation theorem,

$${}_2F_1[a,b;\tfrac{1}{2}+a/2+b/2;1/2]=[\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}+a/2+b/2)]/[\Gamma(\tfrac{1}{2}+a/2)\Gamma(\tfrac{1}{2}+b/2)], \tag{1.6}$$

listed by Slater [2, Appendix III], may be used to sum the inner hypergeometric series on the right of (1.4).

Hence,

$${}_2F_1[a+n,-n;\tfrac{1}{2}+a/2;\tfrac{1}{2}]=[\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}+a/2)]/[\Gamma(\tfrac{1}{2}+a/2+n/2)\Gamma(\tfrac{1}{2}-n/2)]. \tag{1.7}$$

This expression vanishes if n is an odd positive integer. After a little algebra, (1.4) becomes

$$\sum_n [((c_C,n))(a/2,n)(-2x)^n]/[((d_D,n))n!]{}_{C+1}F_D[c_1+n,\dots,c_C+n,a+2n;d_1+n,\dots,d_D+n;x]$$
$$={}_{2C+1}F_{2D}[(c_C/2),\tfrac{1}{2}+(c_C/2),a/2;(d_D/2),\tfrac{1}{2}+(d_D/2);-4^{C-D}x^2]. \tag{1.8}$$

Numerous hypergeometric identities can be deduced from (1.8) by employing, for example, the following hypergeometric summation theorems listed, respectively, by Slater [2, Appendix III, (4), (5), (9), (25), (15)–(18), (11), (26), (13)]:

$${}_2F_1[a,-n;c;1]=(c-a,n)/(c,n) \quad \text{(Vandermonde's theorem)}, \tag{1.9}$$

$$ {}_2F_1[a,b;1+a-b;-1] $$
$$ =[\Gamma(1+a-b)\Gamma(1+a/2)]/[\Gamma(1+a)\Gamma(1+a/2-b)] \quad \text{(Kummer's theorem)}, \tag{1.10} $$

$$ {}_3F_2[a,b,-n;1+a-b,1+a+n;1] $$
$$ =[(1+a,n)(1+a/2-b,n)]/[(1+a/2,n)(1+a-b,n)] \quad \text{(Dixon's theorem)}, \tag{1.11} $$

$$ {}_3F_2[a,1+a/2,-n;a/2,1+a+n;-1]=(1+a,n)/(1/2+a/2,n), \tag{1.12} $$

$$ {}_3F_2[a,1+a/2,-n;a/2,b;1]= [(2+a-b,n)(b-a-1,n)]/[(b,n)(1+a-b,n)], \tag{1.13} $$

$$ {}_3F_2[a,b,-n;1+a-b,1+2b-n;1] $$
$$ =[(a-2b,n)(1+a/2-b,n)(-b,n)]/[(1+a-b,n)(a/2-b,n)(-2b,n)], \tag{1.14} $$

$$ {}_4F_3[a,1+a/2,b,-n;a/2,1+a-b,1+a+n;-1]=(1+a,n)/(1+a-b,n), \tag{1.15} $$

$$ {}_4F_3[a,1+a/2,b,-n;a/2,1+a-b,1+a+n;1] $$
$$ =[(1/2+a/2-b,n)(1+a,n)]/[(1/2+a/2,n)(1+a-b,n)], \tag{1.16} $$

$$ {}_4F_3[a,1+a/2,b,-n;a/2,1+a-b,1+2b-n;1] $$
$$ =[(a-2b,n)(-b,n)]/[(1+a-b,n)(-2b,n)], \tag{1.17} $$

$$ {}_4F_3[a,1+a/2,b,-n;a/2,1+a-b,2+2b-n;1] $$
$$ =[(a-2b-1,n)(1/2+a/2-b,n)(-b-1,n)]/ $$
$$ [(1+a-b,n)(a/2-b-1/2,n)(-2b-1,n)], \tag{1.18} $$

$$ {}_5F_4[a,1+a/2,b,c,-n;a/2,1+a-b,1+a-c,1+a+n;1] $$
$$ =[(1+a,n)(1+a-b-c,n)]/[(1+a-b,n)(1+a-c,n)]. \tag{1.19} $$

2. Identities obtained from (1.9) and (1.19)

Two identities deduced by applying (1.9) and (1.19) to (1.8) are worked out in some detail and other results are stated later without proof.

In (1.8), put $C=D=x=1$, when the inner hypergeometric function on the left becomes

$$ {}_2F_1[c+n,a+2n;d+n;1]. \tag{2.1} $$

For convergence, this series must terminate, so we put

$$ c=-N, \quad N=0,1,2,\ldots \tag{2.2} $$

and apply (1.9). Since $n<N$ (2.1) takes the form

$$(d-a-n,N-n)/(d+n,N-n). \tag{2.3}$$

This expression can be reduced to

$$(d-a,N)/(d,N)\times[(1+a-d,n)(d,n)(-1)^n]/(1+a-d-N,2n) \tag{2.4}$$

by appealing to the usual properties of the Pochhammer symbol. Insert (2.4) into (1.8) with the above restrictions and after a little algebra, we see that

$$\begin{aligned}&(d-a,N)/(d,N)\,{}_3F_2[-N,a/2,1+a-d;\tfrac{1}{2}+a/2-d/2-N/2,1+a/2-d/2-N/2;\tfrac{1}{2}]\\&\quad={}_3F_2[-N/2,\tfrac{1}{2}-N/2;\tfrac{1}{2}+a/2\ d/2,\tfrac{1}{2}+d/2;-1].\end{aligned} \tag{2.5}$$

A second example of a hypergeometric transformation is now obtained from (1.19). Let $C=D=4$ and $x=1$, when the inner hypergeometric function on the left-hand side of (1.8) becomes

$${}_5F_4[a+2n,c_1+n,c_2+n,c_3+n,c_4+n;d_1+n,d_2+n,d_3+n,d_4+n;1]. \tag{2.6}$$

Put $c_1=1+a/2$, $c_2=b$, $c_3=c$, $c_4=-N$, $d_1=a/2$, $d_2=1+a-b$, $d_3=1+a-c$ and $d_4=1+a+N$.

Apply (1.19) to (2.6) which can be written as

$$[(1+a+2n,N-n)(1+a-b-c,N-n)]/[(1+a-b+n,N-n)(1+a-c+n,N-n)]$$

$$=[(1+a,N)(1+a-b-c,N)]/[(1+a-b,N)(1+a-c,N)] \tag{2.7}$$

$$\times[(1+a+N,n)(1+a-b,n)(1+a-c,n)]/[(1+a,2n)(b+c-a-N,n)](-1)^n. \tag{2.8}$$

On substituting these results into (1.8), after a little algebra, we have the expression

$$\begin{aligned}&[(1+a,N)(1+a-b-c,N)]/[(1+a-b,N)(1+a-c,N)]\\&\quad\times{}_3F_2[b,c,-N;\tfrac{1}{2}+a/2,b+c-a-N;\tfrac{1}{2}]\\&={}_8F_7[1+a/4,b/2,1/2+b/2,c/2,1/2+c/2,a/2,-N/2,1/2-N/2;\\&\quad a/4,1/2+a/2-b/2,1+a/2-b/2,1/2+a/2-c/2,1+a/2-c/2,\\&\quad 1+a/2+N/2,1/2+a/2+N/2;-1].\end{aligned} \tag{2.9}$$

3. Further hypergeometric identities

The following results are obtained similarly and are stated without proof:
from (1.10),

$$\begin{aligned}&(1+a,N)/(1+a/2)\,{}_2F_1[a/2,-N;\tfrac{1}{2}+a/2;-\tfrac{1}{2}]\\&\quad={}_3F_2[a/2,-N/2,1/2-N/2;1/2+a/2+N/2,1+a/2+N/2;-1],\end{aligned} \tag{3.1}$$

from (1.11),

$$[(1+a,N)(1+a/2-b,N)/(1+a/2,N)(1+a-b,N)]$$
$$\times {}_3F_2[b,a/2,-N;1/2+a/2,b-a/2-N;1/2]$$
$$= {}_5F_4[b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$1/2+a/2-b/2,1/2+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1], \tag{3.2}$$

from (1.12),

$$(1+a,N)/(\tfrac{1}{2}+a/2,N)\times {}_1F_0[-N;-;(\tfrac{1}{2})^N] = (1+a,N)/[2^N(\tfrac{1}{2}+a/2,N)]$$
$$= {}_4F_3[1+a/4,a/2,-N/2,1/2-N/2;a/4,1/2+a/2+N/2,1+a/2+N/2;-1], \tag{3.3}$$

from (1.13),

$$[(2+a-d,N)(d-a-1,N)]/[(d,N)(1+a-d,N)]$$
$$\times {}_3F_2[1+a/2,1+a-d,-N;1+a/2-d/2-N/2,3/2+a/2-d/2-N/2;1/2]$$
$$= {}_5F_4[1+a/4,a/2,a/2+1/2,-N/2,1/2-N/2;a/4,d/2,d/2+1/2,1/2+a/2;-1], \tag{3.4}$$

from (1.14),

$$[(a-2b,N)(1+a/2-b,N)(-b,N)]/[(a/2-b,N)(-2b,N)(1+a-b,N)]$$
$$\times {}_5F_4[b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$1/2+a/2-b/2,1+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1],$$
$$= {}_5F_4[b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$1/2+a/2-b/2,1+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1], \tag{3.5}$$

from (1.15),

$$(1+a,N)/(1+a-b,N)\times {}_2F_1[b,-N;\tfrac{1}{2}+a/2;1/2]$$
$$= {}_6F_5[1+a/4,b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$a/4,1/2+a/2-b/2,1+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1], \tag{3.6}$$

from (1.16),

$$[(1/2+a/2-b,N)(1+a,N)]/[(a/2+1/2,N)(1+a-b,N)]$$
$$\times {}_2F_1[b,-N;1/2+b-a/2-N;1/2]$$
$$= {}_6F_5[1+a/4,b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$a/4,1/2+a/2-b/2,1+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1], \tag{3.7}$$

from (1.17),

$$[(a-2b,N)(-b,N)]/[(1+a-b,N)(-2b,N)]$$
$$\times\, {}_6F_5[1+a/2,b,1/3+2b/3-N/3,2/3+2b/3-N/3,1+2b/3-N/3,-N;$$
$$1+2b+N,1-a+2b-N,\tfrac{1}{2}+2b-N/2,1+b/2-N/2,\tfrac{1}{2}+b;27/8]$$
$$={}_6F_5[1+a/4,b/2,1/2+b/2,a/2,-N/2,1/2-N/2;$$
$$a/4,1/2+a/2-b/2,1+a/2-b/2,1/2+b+N/2,1+b+N/2;-1], \tag{3.8}$$

and from (1.18),

$$[(a-2b,N)(1/2+a/2-b,N)(-b-1,N)]/[(1+a-b,N)(a/2-1/2-b,N)(-2b-1,N)]$$
$$\times\, {}_8F_7[1+a/2,b,2+b,3/2-a/2+b-N,$$
$$2/3+2b/3-N/3,1+2b/3-N/3,4/3+2b/3-N/3,-N;3+2b,2+2b-a-N,$$
$$1/2+b-a/2-N,1+b/2-N/2,3/2+b/2-N/2,1+b,3/2+b;27/8]$$
$$={}_6F_5[1+a/4,b/2,b/2+1/2,a/2,-N/2,1/2-N/2;$$
$$a/4,1/2+a/2-b/2,1+a/2-b/2,1/2+a/2+N/2,1+a/2+N/2;-1]. \tag{3.9}$$

References

[1] H. Exton, Multiple Hypergeometric Functions, Halsted, New York, 1976.
[2] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, London, 1966.